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Dynamically stable bound states of line defects in biaxial nematics and superfluids

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Abstract

We show that two line defects in biaxial nematics may form a dynamically stable bound state if they are linked, almost parallel to each other and repelling. Such a situation may take place for line defects of certain topology. It occurs also for almost parallel vortices in superfluids and superconductors.

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1. Introduction

In the last 30 years, topology has demonstrated its advantage in the theory of condensed matter. The main application of topology is the classification of thermodynamic phases and defects in ordered media such as liquid crystals and superfluid ^{3}He [1–3]. However, more complicated configurations such as linking defects have not yet been fully investigated. Such configurations are of importance in such interesting physical phenomena as weak turbulence in superfluid helium and liquid crystals, and spinodal decomposition into ordered phase in the same systems [4, 5]. Even less is known about the dynamical stability of these configurations. In this paper, we prove the existence of linked defects in biaxial nematics, the liquid crystal with the complicated order parameter. We point out that a stable configuration of this type appears as an asymptotic solution (at $t \to \infty$) of the Ginzburg–Landau (GL) equations, and hence can be considered as a bound state of two line defects.

The proposed method is quite general and based on the analysis of the GL equations and topology of the order parameter of a corresponding system. The existence of such bound defects in other systems, such as cholesterics, blue phases, superconductors and superfluids, depends on physical conditions which can or cannot be realized in the concrete medium. We leave this discussion to the end of the paper.

2. General model

We start our investigation from biaxial nematics, assuming that our approach of constructing bound states of linked defects may be applicable also to other systems.

2.1. Main equations and definitions

The simplest form of the free energy of biaxial nematics can be written as follows [2, 6],

$$\mathcal{F} = \int_V (\Phi(\mathrm{Tr}[A^2], \mathrm{Tr}[A^3]) + \gamma_1 \partial_r A_{qp} \partial_r A_{qp} + \gamma_2 \partial_q A_{rp} \partial_p A_{rq} + \gamma_3 \partial_q A_{rq} \partial_p A_{rp})\, \mathrm{d}V, \tag{1}$$

where A_{pq} is the order parameter describing a biaxial nematic, that is a real, symmetric-traceless matrix (3×3) [1, 2, 6]. The Latin indices p, q, etc label the Cartesian coordinates on $V \subset \mathbb{R}^3$ with the usual sum over repeated indices. The first term under the integral represents the density of 'potential' energy and the next three terms represent the energy of deformation. Here γ_1, γ_2 and γ_3 are elastic constants.

The density of potential energy Φ in equation (1) attains its minimum at an orbit M of $SO(3)$ action:

$$g : A \Rightarrow gAg^{-1}, \qquad g \in SO(3).$$

The orbit M consists of real traceless symmetric matrices $A = \|A_{pq}\|$ with all eigenvalues being different. It is isomorphic to the coset space $SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq SU(2)/Q$.

Possible types of line defects in the biaxial nematics are determined by the fundamental group π_1 of the order parameter space [1, 2], which were firstly described in [7].

$$\pi_1(M) = Q. \tag{2}$$

Here Q is the unit quaternion group:

$$Q = \{e, -e, i, -i, j, -j, k, -k\}, \tag{3}$$

where

$$i^2 = j^2 = k^2 = -e \qquad i = jk = -kj \text{ etc.}$$

The set of topological charges of line defects or topological types of the defects is the set composed of elements of free homotopy group of the order parameter space (see for example [1, 2]). The free homotopy group is the set of conjugating classes of the based homotopy group π_1. Thus there are four different topological types of line defects in biaxial nematics. They correspond to the conjugating classes of $-e$, $\pm i$, $\pm j$ and $\pm k$ of the quaternion group, respectively. The element e corresponds to a nonsingular solution.

We assume for the sake of simplicity that there is only one stable structure of line defects of each topological type. In this case a defect is determined uniquely by its topological type.

2.2. Bound states of almost parallel line defects

2.2.1. Separate line defects. Let us consider a single line defect in a solid torus $S'^1 \times D^2$, where S'^1 is a circle and D^2 is a disc. Thus $\vec{r} \in \{(x, y, z \bmod L_z) : x^2 + y^2 < L_\perp^2\}$. We assume 'slipping' boundary conditions at $x^2 + y^2 = L_\perp^2$: $\partial_n A_{pq} = 0$. Let the defect be a straight line parallel to the z-axis and placed at $x = y = 0$. Assume that $L_\perp \sim L_z = L_\parallel$, and that the crosswise size of the core of the line defect r_c, determined by the elastic constants γ_α and the 'curvature' of the potential Φ at the minimum, is much less than $L_\perp$. Then out of the core the order parameter, A_{pq} belongs to a small vicinity (in the sense of the Cartan–Killing

metric) of the orbit M, where Φ attains its minimum. The eigenvalues of matrices belonging to the orbit M may be parametrized by the following way,

$$\tfrac{2}{3}S, \qquad -\tfrac{1}{3}(S-3T), \qquad -\tfrac{1}{3}(S+3T),$$

where S and T are, respectively, the uniaxial and biaxial order parameters, and $S > 3T > 0$. Thus the free energy density is reduced to the deformation energy only[1],

$$F_{\text{grad}} = \gamma_1 \partial_r A_{qp} \partial_r A_{qp} + \gamma_2 \partial_q A_{rp} \partial_p A_{rq} + \gamma_3 \partial_q A_{rq} \partial_p A_{rp}, \tag{4}$$

providing that the matrix-order parameter A_{pq} belongs now to the orbit M. The matrix-order parameter depends only on the angle φ in the (x, y)-plane at $r_c \ll r \ll L_\perp$. Hence $F_{\text{grad}} \approx f(\varphi)/r^2$ at $r_c \ll r \ll L_\perp$. Therefore, the integration of F_{grad} over the volume V can be divided into multiple integrals over φ, z and r, that is cylindrical coordinates connected with the defect. The integration over φ at $r_c \ll r \ll L_\perp$ gives some constant depending of the parameters of the model (S, T, γ_i) and the type of defect, and the integration over z gives the length of defect $L_\parallel$ as a factor. The integral over r diverges logarithmically at small and large r at $r_c \ll r \ll L_\perp$. Thus the free energy of the line defect of type $\lambda \in \{-e, i, j, k\}$ can be presented as

$$\mathcal{F}_\lambda = K_\lambda L_\parallel \ln L_\perp / r_c \left(1 + \mathcal{O}\left(\frac{1}{\ln L_\perp / r_c}\right)\right), \tag{5}$$

where

$$K_\lambda = \gamma_1 S^2 \bar{K}_\lambda \left(T/S, \gamma_2/\gamma_1, \gamma_3/\gamma_1\right). \tag{6}$$

2.2.2. Two linked line defects. Consider now a system of two almost parallel line defects η_1 and η_2 of types λ_1 and λ_2. Assume that angles between tangent lines of η_1 and η_2 and the z-axis is small ($\ll \pi/2$) and that the distance between the defects in the (x, y) planes obeys the condition $r_c \ll r \ll L_\perp$ in each cross-section perpendicular to the z-axis. Besides, we suppose that the dependence on z is smooth enough. Assuming also, that

$$\lambda_1 \circ \lambda_2 = \lambda_3, \tag{7}$$

we may write

$$\mathcal{F}_{12} = \bar{\mathcal{F}}_{12} + \gamma_1 S^2 \mathcal{O}(1) \tag{8}$$

at $r_c/r \to 0$ and $r/L_\perp \to 0$, where

$$\bar{\mathcal{F}}_{12} = \gamma_1 S^2 \left[\bar{K}_{\lambda_1} \int_{z=0}^{L_\parallel} \ln \frac{r(z)}{r_c} \, \mathrm{d}l_1 + \bar{K}_{\lambda_2} \int_{z=0}^{L_\parallel} \ln \frac{r(z)}{r_c} \, \mathrm{d}l_2 + \bar{K}_{\lambda_3} \int_{z=0}^{L_\parallel} \ln \frac{L_\perp}{r(z)} \, \mathrm{d}z \right]. \tag{9}$$

To obtain this equation, we assumed that the order parameter obeys the exact GL equation in the region out of the cores of the defects that provides minimum of the free energy in this region. In this case the free energy of the whole system becomes the functions of position of the defects with a relative accuracy $\mathcal{O}(1/\ln(r/r_c))$. Just this expression for the functional of free energy is presented in equation (9). It was derived analogously to equation (5).

Suppose that the constants $\bar{K}_{\lambda_1}$, $\bar{K}_{\lambda_2}$ and $\bar{K}_{\lambda_3}$ obey the following opposite 'triangle' inequality:

$$\bar{K}_{\lambda_3} > \bar{K}_{\lambda_1} + \bar{K}_{\lambda_2}. \tag{10}$$

[1] General form of the deformation enegry for biaxial nematics may be found in [8]. The subsequent consideration, excluding of course section 3, does not depend on the concrete form of the deformation energy.

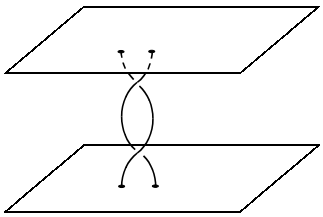

Figure 1. Example of two linked line defects with $n = 1$.

It means that line defects repel each other. To clarify this fact we may consider two parallel straight defects. Then equation (10) gives

$$\bar{\mathcal{F}}_{12} = \gamma_1 S^2 L_\parallel [\bar{K}_3 \ln L_\perp / r_c + (-\bar{K}_3 + \bar{K}_1 + \bar{K}_2) \ln r / r_c],$$

where r is the distance between the defects, and the shortcuts $\bar{K}_{\lambda_1} = \bar{K}_1, \ldots$ are introduced here and below if it does not lead to confusion. The latter equation means that the free energy decreases when the distance between the defects increases that corresponds to repulsion.

If line defects are not linked, the free energy equation (9) reaches its minimum when the line defects are parallel to the z-axis and the distance between them is of the order of $L_\perp/2$.

Assume now that line defects are linked, and the linking number is equal to $n > 0$. In this case increasing the distance between the defects leads to a considerable increase in the lengths of line defects, and hence to a growth of the free energy. At sufficiently small distances between the defects, the repulsion of the two defects dominates. It means that there is a minimum of the free energy for a certain distance between defects.

Now we prove our main statement that the minimum is attained for the state when line defects have the form of helices of certain radii r_1 and r_2, respectively, nested in each other. See figure 1 as an example for $n = 1$.

Assume that the helices have the same axis and the same pitch and turned about 180° around each other. In this state, $\bar{\mathcal{F}}_{12}$ can be evaluated at $r_{1,2}/L_\parallel \to 0$ as

$$\bar{\mathcal{F}}_{12} = \tilde{\mathcal{F}}_{12} + \gamma_1 S^2 \mathcal{O}\left(\frac{r_{1,2}^4}{L_\parallel^4} \ln \frac{r_{1,2}}{r_c}\right),$$

where

$$\tilde{\mathcal{F}}_{12} = \gamma_1 S^2 L_\parallel \left\{(\bar{K}_1 + \bar{K}_2) \ln \frac{r_1 + r_2}{r_c} - \bar{K}_3 \ln \frac{r_1 + r_2}{L_\perp} + 2\pi^2 n^2 \left(\bar{K}_1 \frac{r_1^2}{L_\parallel^2} + \bar{K}_2 \frac{r_2^2}{L_\parallel^2}\right) \ln \frac{r_1 + r_2}{r_c}\right\}, \tag{11}$$

where $\tilde{\mathcal{F}}_{12}$ depends only on r_1 and r_2. To find a minimum of expression (11) we differentiate it with respect to r_1 and r_2. It follows from a straightforward calculation[2] that the minimum is achieved at $r_1 = \tilde{r}_1$ and $r_2 = \tilde{r}_2$, whereas the latter values are defined by the expressions,

$$\tilde{r}_1 = \bar{r}_1 \left(1 + o\left(\frac{1}{\Lambda^{1-\alpha}}\right)\right), \tag{12}$$

$$\tilde{r}_2 = \bar{r}_2 \left(1 + o\left(\frac{1}{\Lambda^{1-\alpha}}\right)\right), \tag{13}$$

[2] Some details are placed in the appendix.

where α is any number $0 < \alpha < 1$, and

$$\bar{r}_1 = \frac{L_\parallel}{2\pi n\sqrt{\Lambda}}\sqrt{\frac{\Delta\bar{K}_{123}\bar{K}_2}{\bar{K}_1(\bar{K}_1+\bar{K}_2)}} \tag{14}$$

$$\bar{r}_2 = \frac{L_\parallel}{2\pi n\sqrt{\Lambda}}\sqrt{\frac{\Delta\bar{K}_{123}\bar{K}_1}{\bar{K}_2(\bar{K}_1+\bar{K}_2)}}, \tag{15}$$

$$\Lambda = \ln\left(\frac{L_\parallel}{2\pi n r_c}\sqrt{\frac{\Delta\bar{K}_{123}(\bar{K}_1+\bar{K}_2)}{\bar{K}_1\bar{K}_2}}\right), \tag{16}$$

and

$$\Delta\bar{K}_{123} = \bar{K}_3 - \bar{K}_2 - \bar{K}_1. \tag{17}$$

These estimates are valid when $r_1, r_2 \ll L_\parallel$, implying that $2\pi n\sqrt{\Lambda} \gg 1$.

This bound state of linked defects provides the local minimum of the free energy. To prove this statement, we calculate the second variation of the functional (9) (in the reduced form of equation (9). The first one vanishes due to the definition of the linked state. Thus

$$\delta^2\bar{\mathcal{F}}_{12} = \int_0^{L_\parallel}\delta^2\bar{F}_{12}(z)\,\mathrm{d}z, \tag{18}$$

where

$$\delta^2\bar{F}_{12}(z) = \gamma_1 S^2\Bigg\{A_1(\delta r_1+\delta r_2)^2 + A_2(\delta r_1)^2 + A_3(\delta r_2)^2 + A_4\left(\frac{\mathrm{d}}{\mathrm{d}z}\delta r_1\right)^2 + A_5\bar{r}_1^2\left(\frac{\mathrm{d}}{\mathrm{d}z}\delta\varphi_1\right)^2 + A_6\left(\frac{\mathrm{d}}{\mathrm{d}z}\delta r_2\right)^2 + A_7\bar{r}_2^2\left(\frac{\mathrm{d}}{\mathrm{d}z}\delta\varphi_2\right)^2\Bigg\}. \tag{19}$$

Here the spatial position of the defects is parametrized, using the cylindrical coordinate system (r, φ, z), in the following way: $(r_{1,2}(z), \varphi_{1,2}(z), z)$, where

$$r_{1,2}(z) = \tilde{r}_{1,2} + \delta r_{1,2}(z), \tag{20}$$

$$\varphi_{1,2}(z) = \pi\Delta_{1,2} + 2\pi n\frac{z}{L_\parallel} + \delta\varphi_{1,2}(z), \qquad \Delta_1 = 0 \quad \text{and} \quad \Delta_2 = 1, \tag{21}$$

where $\tilde{r}_{1,2}$ and the first two terms in equation (20) describe the equilibrium form of the two linked defects. See equations (12) and (13) and explanations above them. The coefficients $A_1, \ldots, A_7$ entering equation (19) are positive (if $\Delta K_{\lambda_1,\lambda_2,\lambda_3} > 0$) and can be expressed as follows:

$$A_1 = \frac{\Delta K_{\lambda_1,\lambda_2,\lambda_3}}{2(\bar{r}_1+\bar{r}_2)^2}\left(1 + o\left(\frac{1}{\Lambda^{1-\alpha}}\right)\right), \tag{22}$$

$$A_{2,3} = \pi^2 n^2\frac{K_{\lambda_{1,2}}}{L_\parallel^2}\Lambda\left(1 + o\left(\frac{1}{\Lambda^{1-\alpha}}\right)\right), \tag{23}$$

$$A_{4,\ldots,7} = \frac{1}{2}\left(1 + o\left(\frac{1}{\Lambda^{1-\alpha}}\right)\right). \tag{24}$$

Here α is any $0 < \alpha < 1$.

We point out that the absence of other possible terms in equation (19), for example such as $\delta\varphi_{1,2}\delta r_{1,2}$, follows from the straightforward calculations and the spatial symmetry of the system. The latter one means that functional (9) is invariant under transformations such as reflections, translations and rotations in our solid torus, and that the equilibrium state of *the two line defects* is invariant under the helical symmetry. This symmetry includes a half-turn around a certain axis perpendicular to the axis of the two-helical state. The first derivatives of the free energy (9) with respect to $\varphi_{1,2}$ vanish at the helical state for the same reason.

Recall that equations (20) and (21) imply general though smooth enough perturbations of the helical form of the two linked defects described above.

Thus we may see that the steady state of two linked line defects corresponds to the local minimum of the free energy. It means that the steady state is stable at least locally.

3. Verification of the opposite triangle condition for a model of the biaxial nematics

In this section, we checked the opposite 'triangle' inequalities: $\bar{K}_{\lambda_1} + \bar{K}_{\lambda_2} < \bar{K}_{\lambda_1 \circ \lambda_2}$ for the simplest possible case of the elasticity model.

The constants $\bar{K}_\lambda$ are calculated for a particular case when

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma.$$

In this case (see [9, 5])

$$\begin{aligned} &\bar{K}_i = (1 + T/S)^2, && \bar{K}_j = (1 - T/S)^2, \\ &\bar{K}_k = 4T^2/S^2, && \bar{K}_{-e} = 16T^2/S^2. \end{aligned} \tag{25}$$

We see that

$$\bar{K}_i > \bar{K}_j > \bar{K}_k, \tag{26}$$

and

$$\begin{aligned} &\bar{K}_k + \bar{K}_k < \bar{K}_{-e}, && \bar{K}_k + \bar{K}_j < \bar{K}_i, \\ &\bar{K}_k + \bar{K}_i > \bar{K}_j, && \bar{K}_j + \bar{K}_i > \bar{K}_k, \\ &\bar{K}_i + \bar{K}_i > \bar{K}_{-e}, \end{aligned} \tag{27}$$

for all possible ranges of the parameter T/S: $S/3 > T > 0$. For the pair of two j defects we have

$$\bar{K}_j + \bar{K}_j > \bar{K}_{-e} \qquad \text{when} \qquad \frac{T}{S} < \frac{2\sqrt{2} - 1}{7}, \tag{28}$$

$$\bar{K}_j + \bar{K}_j < \bar{K}_{-e} \qquad \text{when} \qquad \frac{T}{S} > \frac{2\sqrt{2} - 1}{7}. \tag{29}$$

These inequalities mean that only pairs of line defects with $(\lambda_1, \lambda_2) = (k, k)$ and $(\lambda_1, \lambda_2) = (k, j)$ form the stable bound states in the model of biaxial nematics considered in this section when $T/S < (2\sqrt{2} - 1)/7$. Besides, these two pairs of defects the stable linked state may form the pair $(\lambda_1, \lambda_2) = (j, j)$ when $(2\sqrt{2} - 1)/7 < T/S < 1/3$.

Using the standard arguments of general position, it is possible to prove that our conclusions remain valid in a vicinity of the space of parameters (γ_i) which are not necessarily equal.

4. Discussion and conclusions

Linked bound states of line defects described in this paper may be important to condensed matter, since they may exist in asymptotic relaxed state of the system.

We considered the linked defects inside the solid torus $S'^1 \times D^2$ that corresponds to the periodic boundary conditions along the defects. The effect considered in the paper can take place also for more general boundary conditions, when pinning of the line defects is possible. Methods developed in the previous sections may be applied to other systems.

4.1. Uniaxial nematics

In this case, the order parameter space is $M = RP^2$ and $\pi_1(M) = \mathbb{Z}_2$. Hence the combination of two defects results in a nonsingular state, whereas the defects attract each other. Hence, there is no bound linked state of two defects of the type considered above.

4.2. Cholesterics

There are natural candidates for the existence of the bound linked states of two defects since they have globally the same order parameter [2] as biaxial nematics. However, the situation is more subtle because locally the order parameter is the same as in the uniaxial nematics. Unlike nematics, the state of cholesterics is not fixed by the director ($\vec{n}$) which varies in space and describes a spiral curve. As a result the elastic properties of cholesterics is highly anisotropic, and singularities in the director field are the most energetically unfavourable. The line defect types may be chosen so that the effective elastic constants for the line defects of the conjugation classes $-e$ and i are considerably less than the effective elastic constant for the conjugation classes j and k, with the director field having no singularities for the former two classes. Such a situation is described sometimes in terms of 'double topology' [2]. It is natural to conjecture that these properties of cholesterics lead to the existence of bound linked states of two line defects of type i and two line defects of types i and j (or k). We will consider this problem in the forthcoming, detailed paper. Finally, we mention some other systems where the configuration of linked defects can exist.

4.3. Superfluid ^{4}He

Here the order parameter is a complex function $\psi = |\psi| e^{i\varphi}$, where ψ as a function of the space coordinates is governed by the GL equation. The space of the order parameter is equal to S^1 with $\pi_1(S^1) = \mathbb{Z}$. In this case the existence of vortices is well known. The opposite 'triangle' condition is obviously valid for vortices with almost parallel vorticities, and hence linked vortices may exist. In a rotating vessel, the vortex lattice transforms into a dynamical system of linked lines and this state corresponds to a turbulent regime. This regime is widely discussed in the current literature [14, 15].

4.4. Superfluid ^{3}He

There is a lot of different line defects in superfluid ^{3}He depending on the phase (A or B) and on the strength of spin–orbital interaction [10, 11]. Some of them may form bound linked states.

4.5. Blue phase

The next example is a specific class of cholesteric blue phases. Here we encounter a new phenomenon. To study blue phases it is necessary to take into account the surface term in the density of potential energy.

4.6. Unconventional superconductivity

Our last example is recently discovered unconventional superconducting states of Sr_2RuO_4 and $PrOs_4Sb_{12}$ [12]. It has been proposed that the superconducting order parameter is a triplet in both cases. An analogy with superfluid phases in ^{3}He looks quite natural. So it is possible to propose the existence of similar vortex structure in these systems. For example, half-quantum vortices in Sr_2RuO_4 were discussed in [13].

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Appendix

A few technical details to obtain equations (12)–(16) are considered in this appendix.

After differentiation of the reduced free energy (11) of the state of two nested in each other helical defects described in section 2.2.2 with respect to radii r_1 and r_2 of these helices, we obtain the following two equations for $\tilde{r}_1$ and $\tilde{r}_2$:

$$\left(\Delta\bar{K}_{123} - 2\pi^2 n^2\left(\bar{K}_1 x_1^2 + \bar{K}_2 x_2^2\right)\right)\frac{1}{x_1+x_2} = 4\pi^2 n^2 \bar{K}_1 x_1\left(\ln(x_1+x_2) + \ln\frac{L_\parallel}{r_c}\right), \tag{A.1}$$

$$\left(\Delta\bar{K}_{123} - 2\pi^2 n^2\left(\bar{K}_1 x_1^2 + \bar{K}_2 x_2^2\right)\right)\frac{1}{x_1+x_2} = 4\pi^2 n^2 \bar{K}_2 x_2\left(\ln(x_1+x_2) + \ln\frac{L_\parallel}{r_c}\right), \tag{A.2}$$

where $x_1 = \tilde{r}_1/L_\parallel$ and $x_2 = \tilde{r}_2/L_\parallel$. The variables x_1 and x_2 enter symmetrically on the left-hand side of these equations and in the brackets on the right-hand side of them. It gives that

$$x_2 = x_1\frac{\bar{K}_1}{\bar{K}_2}, \tag{A.3}$$

and

$$\bar{K}_2\Delta\bar{K}_{123} = 2\pi^2 n^2 x_1^2\left(\left(\bar{K}_1^2 + \bar{K}_1^2\right) + 2\bar{K}_1(\bar{K}_1+\bar{K}_2)\left(\ln x_1 + \ln\frac{L_\parallel(\bar{K}_1+\bar{K}_2)}{r_c\bar{K}_2}\right)\right). \tag{A.4}$$

The latter equation contains the large parameter $\ln L_\parallel/r_c$ when $r_c/L_\parallel \to 0$. All other parameters entering this equation are of the order of 1. It means that $x_1 \propto 1/\sqrt{\ln L_\parallel/r_c}$ at $r_c/L_\parallel \to 0$ and tends to 0 at $r_c/L_\parallel \to 0$. Nevertheless $|\ln x_1|/\ln(L_\parallel/r_c) \propto \ln\ln(L_\parallel/r_c)/\ln(L_\parallel/r_c) \to 0$ at $r_c/L_\parallel \to 0$. It means that equation (A.4) can be solved iteratively using the fact that the absolute value of the first logarithm on the right-hand side of this equation is much less than the second logarithm there. The leading term of the asymptotics

$x_1 = \bar{x}_1(1+o(\ln^{\alpha-1}(L_\parallel/r_c))$ at $r_c/L_\parallel \to 0$ for any $0 < \alpha < 1$ may be obtained by substituting this expression into equation (A.4) and neglecting minor terms:

$$\bar{K}_2 \Delta \bar{K}_{123} = 4\pi^2 n^2 \bar{K}_1 (\bar{K}_1 + \bar{K}_2) \bar{x}_1^2 \ln(L_\parallel/r_c).$$

These arguments give asymptotics (12)–(16). We pay attention to the fact that a numerical factor of the order of 1 under the logarithm in the definition (16) has no significant sense and was introduced for convenience only.

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